CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of this talk is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor and, as a app., the two routes to reach the free algebra
    2. without functor sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

Below a quick list of the categories of use in combinatorics (k is a given field), morphisms are standard.

1. **St**, the category of sets
2. **Mon**, the category of monoids
3. **CMon**, the category of commutative monoids
4. **Gp**, the category of groups
5. **Ring**, the category of rings
6. **CRing**, the category of commutative rings
7. $\mathbf{Vect_k}$, the category of k-vector spaces
8. $\mathbf{Lie_k}$, the category of k-Lie algebras
9. $\mathbf{AAU_k}$, the category of k-Associative Algebras with Unit
10. $\mathbf{CAAU_k}$, the category of k-Associative and Commutative Algebras with Unit

Useful categories/2

11. **Mg**, the category of Magmas i.e. sets with only a binary law (without conditions)
12. $\mathbf{Alg_k}$, the category of k-Algebras (without conditions)
13. $\mathbf{DiffAlg_k}$, the category of k-Associative Differential Algebras with Unit.
14. $\mathbf{CDiffAlg_k}$, the category of k-Associative Commutative Differential Algebras with Unit.
15. **DiffRing**, the category of Differential rings.
16. **CDiffRing**, the category of Commutative Differential Rings.
17. $S - \mathbf{grRing}$, the category of S-graded rings.
18. $S - \mathbf{grAlg_k}$, the category of the category of S-graded k-Algebras (without conditions).

Remarks

i) All of these have a standard forgetful functor to **St**. They usually compose and factor nicely. See also [20].
ii) For $\mathbf{k} = \mathbb{Z}$, one has

$$\mathbf{DiffAlg}_{\mathbb{Z}} = \mathbf{DiffRing} \text{ and } \mathbf{CDiffAlg}_{\mathbb{Z}} = \mathbf{CDiffRing}.$$

iii) As usual all will be numbered to facilitate reference in case of interaction.

Graded rings/algebras

Definition (gradation)

Let **k** be a ring, R be a **k**-algebra and S be a semigroup. We will say that $R \in S - \mathbf{grAlg_k}$ (a S-graded **k**-algebra) iff

1. $R = \oplus_{s \in S} R_s$
2. for all $s, t \in S$, $R_s.R_t \subset R_{s.t}$

Comments

i) This definition, from wikipedia [22] and recent (last revised 3 Feb. 2021), so large because it does not suppose the monoid of degrees to be commutative, is a good opportunity for us because it allows elegance. Let us see how it operates.
ii) The example $T(M)$ below shows that condition 2 is not reducible to the commutative case (we can easily Taylor counterexamples with $R_{s.t} \neq R_{t.s}$).

First ingredient: gradation of $T(M)/1$.

1. Let M be a **k**-module, then $T(M)$ is the algebra

$$\mathbf{k}.1_{T(M)} \oplus M \oplus M^{\otimes 2} \oplus \underbrace{M^{\otimes 3}}_{=T^3(M)} \oplus M^{\otimes 4} \oplus \cdots$$

with the law given by blunt concatenation
$conc: \ T^p(M) \times T^q(M) \to T^{p+q}(M)$ i.e.

$$conc: \ (m_1 \otimes \cdots \otimes m_p) \times (m_{p+1} \otimes \cdots \otimes m_{p+q}) := m_1 \otimes \cdots \otimes m_{p+q}$$

one checks at once that it is bilinear and then *conc* (we use the same identifier with a slight abuse of language) defines a law of algebras

$$conc: \ T(M) \otimes T(M) \to T(M)$$

associativity of this law in general is the subject of pentagonal and hexagonal conditions, we will return to this in a forthcoming CCRT.

First ingredient: gradation of $T(M)/2$.

2. If $M = M_r \oplus M_b$ (say "red" and "blue" sectors), then $T(M)$ is $\{r, b\}^*$-graded (in the large sense of [22]).
3. Indeed, we will say that a tensor $m_1 \otimes \cdots \otimes m_p$ is of type $w \in \{r, b\}^p \subset \{r, b\}^*$ if, for all $j \in [1, \cdots, p]$ $m_j \in M_{w[j]}$.
4. It is an (easy) exercise of redaction to prove that
   1. One has the direct sum
   $$\oplus_{w \in \{r,b\}} T_w(M) \tag{1}$$
   where $T_w(M)$ is the submodule of tensors of type w.
   2. and $T_u(M).T_v(M) \subset T_{uv}(M)$
5. So is the gradation of $T(M)$ by a free monoid.
6. Similarly, one can prove that if $M = \oplus_{i \in I} M_i$, then $T(M)$ is I^*-graded (please ask for formal or technical exercises).

About MO question 625874

7. In [23], Martin Brandenburg, fully describes the free product of two Noncommutative rings. In his first version the two factors were claimed to embed within the product. This is not true in general, but it is for augmented rings. Let us recall the elements of the problem.

Let us now recall the problem of coproducts in a category (all here is stated within the same category $\mathcal{C}$).

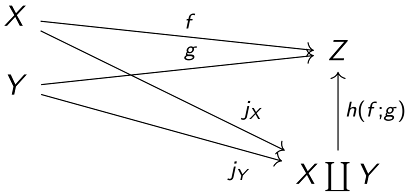


Figure: Coproduct $(j_X, j_Y; X \coprod Y)$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X \coprod Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{2}$$

Coproducts: Augmented $\mathbf{k}-\mathbf{AAU}$

All here is stated within the same category *Augmented* $\mathbf{k}-\mathbf{AAU}$.

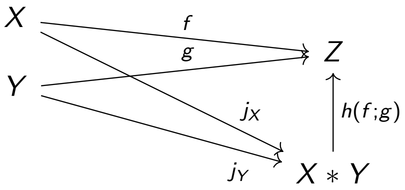


Figure: Coproduct $(j_X, j_Y; X * Y)$ here $h(f;g) = f * g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X*Y,Z)\big)\\&\big(h(f;g)\circ j_X = f \text{ and } h(f;g)\circ j_Y = g\big)\end{aligned} \tag{3}$$

Explicit contruction

8. The category of Augmented $\mathbf{k} - \mathbf{AAU}$ (see [26] and the very particular case [24]) is as follows
    1. **Objects**: Pairs (A, ϵ) where $A \in \mathbf{k} - \mathbf{AAU}$ and $\epsilon \in \Xi(A)$
    2. **Arrows**: $f \in Ar(\mathbf{k} - \mathbf{AAU})$ such that (with $f: \ A_1 \to A_2 \ \epsilon_2 f = \epsilon_1$.
9. With the preceding notations, one begins with two augmented algebras $(X, \epsilon_A), \ (Y, \epsilon_B)$.
10. **Q1**: Construct the universal solution using $T(X_+ \oplus Y_+)$ and the grading in $\{r, b\}^*$ taking $M_r = X_+, \ M_b = Y_+$.

The categories **DiffRing**, **CDiffRing**, $\mathbf{DiffAlg_k}$, $\mathbf{CDiffAlg_k}$

1. We begin with $\mathbf{DiffAlg_k}$
   Let $\mathbf{k}$ be a ring $\mathbf{DiffAlg_k}$ is the category of pairs $(\mathcal{A}, \partial)$ where $\mathcal{A} \in \mathbf{AAU_k}$ and $\partial \in Der(\mathcal{A})$. An arrow $f : (\mathcal{A}, \partial_A) \to (\mathcal{B}, \partial_B)$ is an arrow $f \in Hom_{\mathbf{k}}(\mathcal{A}, \mathcal{B})$ such that $f\partial_A = \partial_B f$.
2. For $(\mathcal{A}, \partial_A) \in \mathbf{DiffAlg_k}$, $ker(\partial_A)$ is a $\mathbf{k}$-subalgebra of $\mathcal{A}$ called that of constants of $\mathcal{A}$.

We now describe the free objects

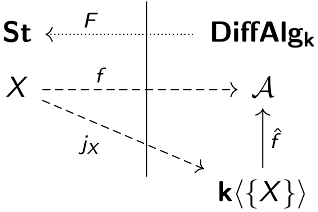
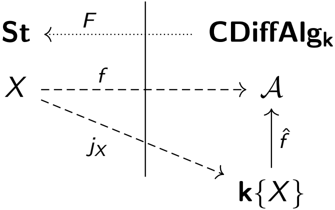


Figure: A solution of the universal problem w.r.t. the natural forgetful functor from $\mathbf{DiffAlg_k}$ to **St**.

Construction of $\mathbf{k}\langle\{X\}\rangle$ and $\mathbf{k}\{X\}$

1. We describe the structure. Let X be an alphabet.
   The free object $\mathbf{k}\langle\{X\}\rangle$ is:
   1. a free algebra $\mathbf{k}\langle X \times \mathbb{N}\rangle$ where, for all $x \in X$, is noted $(x, n) = x^{[n]}$ and, for convenience, $x^{[0]} = x$. This algebra is equipped with the derivation ∂ such that $\partial(x^{[k]}) = x^{[k+1]}$
   2. Existence of ∂ as a derivation is standard (see e.g. [2], Ch I, §2.8 *Extension of derivations*).
   3. The construction is similar to what is to be found in [21], but in the noncommutative realm.
2. We now say a word of the construction in [21]



Construction of $\mathbf{k}\{X\}$

3. Construction of $\mathbf{k}\{X\}$ is very similar to that of $\mathbf{k}\langle\{X\}\rangle$ but
   1. It is devoted to the category $\mathbf{CDiffAlg_k}$ (commutative differential $\mathbf{k}$-algebras)
   2. It uses commutative polynomials i.e. the basic algebra is $\mathbf{k}[X \times \mathbb{N}]$ (and not $\mathbf{k}\langle X \times \mathbb{N}\rangle$) with the same notations ($(x, n) = x^{[n]}$ and $x^{[0]} = x$).
   3. It is the one used for Proposition 2 in Vu's talk (and, in fact, the construction can be done using $\mathbf{k}\{X\}$ with $Y_i^{[j]} = Y_{ij}$ and a suitable ideal).
   4. We recall Proposition 2.

Proposition 2

Let F be a differential field with algebraically closed field of constants C_F and $\mathcal{L}(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0$ be defined over F. Then there exists a Picard-Vessiot extension L of F for $\mathcal{L}$, that is unique up to differential F-isomorphism.

Application: Cartan theorem in Banach algebras (without transversality nor Lipschitz condition)

See [25] for motivation.

Theorem Let $\mathcal{B}$ be a Banach algebra (with unit e) and G be a closed subgroup of $\mathcal{B}^{-1}$ (the group of multiplicative inverses). Let $L(G)$ be the tangent space of G and $m: \ I \to L(G)$ be a continuous function ($I \subset \mathbb{R}$ is an open interval containing $0_{\mathbb{R}}$), then

i) The following system

$$y'(t) = m(t)y(t) \ ; \ y(0) = e$$

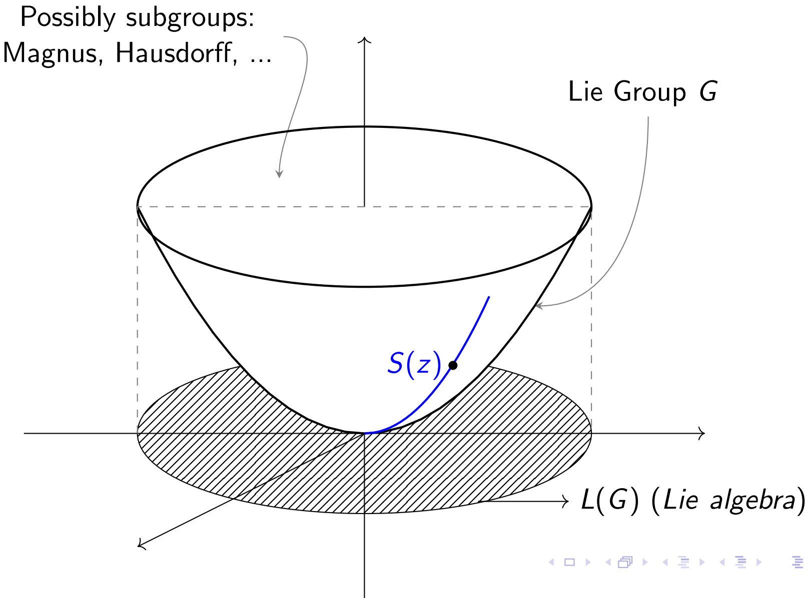
admits a unique solution, say $s(t)$.

ii) The trajectory of s is entirely in G (in other words $t \mapsto s(t)$ is a path drawn on G). My questions are the following:

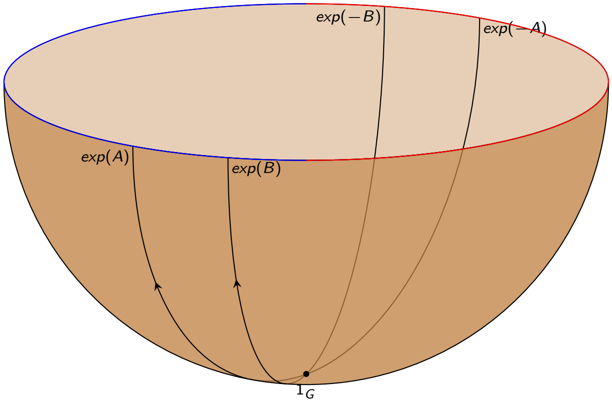
Q1) Is it known? (I expect so, at least of the specialists)

Q2) If yes, is there a sound reference? (not general, but about this very simple and precise property).

Magnus and Ш_φ-Hausdorff groups



Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{\text{Ш}}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

About Magnus expansion and Poincaré-Hausdorff formula/1

Let $(\mathbb{C}\langle\{X\}\rangle, \partial)$ be the differential algebra freely generated by X (a single formal variable). We define a comultiplication Δ by asking that all $X^{[k]}$ be primitive note that Δ commutes with the derivation. Setting, in $\widehat{\mathbb{C}\langle\{X\}\rangle}$, $D = \partial(e^X)e^{-X}$, direct computation shows that D is primitive and hence a Lie series[1], which can therefore be written as a sum of (evaluations of) Dynkin trees.
On the other hand, the formula

$$D = \sum_{k\geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l (\partial X) X^{k-1-l} \cdot \sum_{n\geq 0} \frac{(-X)^n}{n!} \tag{4}$$

suggests that all bidegrees, in $(X, \partial X)$, are of the form $[n, 1]$ and thus, there exists an univariate series $\Phi(Y) = \sum_{n\geq 0} a_n Y^n$ such that $D = \Phi(ad_X)[\partial X]$.

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[1] Which would be trivial,if we were in $\mathbb{C}\{X\}$ (i.e. X commutes with ∂X, as there $D = \partial(X)$, but this is not the case within $\mathbb{C}\langle\{X\}\rangle$ as shows the computation (4).

About Magnus expansion and Poincaré-Hausdorff formula/2

Using left and right multiplications by X (resp. noted g, d), we can rewrite (4) as

$$D = \Big(\sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l} [\partial X] \Big) e^{-X} \tag{5}$$

but, from the fact that g, d commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$) and

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y-Z) + Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y-Z)^j Z^{k-j}$$

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y-Z)+Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y-Z)^j Z^{k-j} \tag{6}$$

Taking notice that $(g - d) = ad_X$ and pluging (6) into (4), one gets

$$D = \Big(\sum_{k\geq 1} \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (ad_X)^{j-1} d^{k-j}[\partial X] \Big) e^{-X} =$$
$$\frac{1}{ad_X} \Big(\sum_{k\geq 1} \sum_{j=1}^{k} \frac{1}{j!(r-j)!} (ad_X)^j d^{k-j}[\partial X] \Big) e^{-X} = \frac{e^{ad_X} - 1}{ad_X}[X'] \tag{7}$$

which is Poincaré-Hausdorff formula (of course $\frac{e^{ad_X} - 1}{ad_X}$ stands for the substitution of ad_X in the formal series corresponding to the entire function $\frac{e^z - 1}{z}$).

Concluding remark and final questions

R1) Contrariwise to Ado's theorem for Lie algebras, some finite-dimensional Lie groups do not admit a finite-dimensional matrix realization (as the metaplectic group for example), but realization as subgroups of some $\mathcal{B}^{-1}$, $\mathcal{B}$ being a Banach algebra (thus infinite dimensional).

Q2) How to adapt this scheme to the group $Mag(\mathbb{C}, X) = 1 + \mathbb{C}_+\langle\langle X\rangle\rangle$ and its $\sqcup\!\sqcup_{\varphi}$-Hausdorff subgroups ? (groups of characters)

Q3) Are there MRS-factorization analogues ?

Thank you for your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

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[2] N. Bourbaki.– *Lie Groups and Lie Algebras, ch 1-3*, Addison-Wesley, ISBN 0-201-00643-X

[3] P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

[4] M. Deneufchâtel, GD, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

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[19] J. Voight, *Quaternion algebras*, `https://math.dartmouth.edu/~jvoight/quat-book.pdf`

[20] Adjuncts in nlab. `https://ncatlab.org/nlab/show/adjunct`

[21] M. van der Put, M. F. Singer.– *Galois Theory of Linear Differential Equations*, Springer (2003)

[22] Graded rings, see "Graded Rings and Algebras" in
https://en.wikipedia.org/wiki/Graded_ring

[23] How to construct the coproduct of two non-commutative rings
https://math.stackexchange.com/questions/625874

[24] Definition of (commutative) free augmented algebras
https://mathoverflow.net/questions/352726

[25] Closed subgroup (Cartan) theorem without transversality nor Lipschitz condition within Banach algebras
https://mathoverflow.net/questions/356531

[26] Definition of augmented algebras (general)
https://ncatlab.org/nlab/show/augmented+algebra